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## A Note on Schönemann's Refutation of Spearman's Hypothesis

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Spearman's hypothesis states that there is a positive relationship between the standardized black-white differences in means on cognitive tests and the loadings of these tests on the general common factor identified as general cognitive ability. Schönemann (1992) claims to demonstrate that, within an analysis of principal components, this relationship is ascribable to a statistical artifact. In the present note, Schönemann's refutation is shown to be incorrect.

### *Introduction*

Spearman's hypothesis, as presented by Jensen (1985), states that the relative magnitudes of the standardized differences in means between blacks and whites on a wide variety of cognitive tests are related predominantly to the relative magnitudes of the tests' loadings on a common factor. This common factor is interpreted as general cognitive ability and is denoted *g*. Jensen reported a Spearman's rho of .59 based on 11 studies. In a second study of small, but well-matched samples, Spearman's rho was found to equal .75 (Naglieri & Jensen, 1987). Additional support for the hypothesis is presented in Jensen (1987) and Jensen (1993).

Schönemann claims that the positive correlation predicted by Spearman's hypothesis is ascribable to a statistical artifact. Specifically, Schönemann (1989; 1992) purports to show that, in an analysis of principal components, a perfect positive relationship is a mathematical necessity. The theorem, upon which Schönemann's bases his claim, appears to have gone unscrutinized. It has however been pointed out that the general absence of perfect, or near perfect, correlations suggests that Schönemann's refutation is

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incorrect (Roskam & Ellis, 1992). Similarly, Braden (1989) has demonstrated, in an analysis of intelligence test scores of normal-hearing and deaf children, the absence of a positive correlation between differences in means and test's  $g$  loadings. Braden's findings are hard to reconcile with the notion that the positive relationship is the necessary result of a statistical artifact.

Rather than presenting indirect empirical evidence, we show that the theorem, presented by Schönemann (1989, 1992) in support of his claim, is incorrect.

### *Schönemann's Theorem*

To show that Spearman's hypothesis is attributable to mathematical necessity, Schönemann (1992) introduces the following theorem:

If the range  $Re^p$  of a  $p$ -variate normal random vector  $\mathbf{y} \sim N_p(0, \Sigma)$  is partitioned into a High set (H) and a Low set (L) by the plane  $\Sigma_k \mathbf{y}_k = 0$ , and both within covariance matrices  $\Sigma_H, \Sigma_L$  remain positive, then (a) the mean difference vector  $\mathbf{d} = E(\mathbf{y}|H) - E(\mathbf{y}|L)$  is collinear with the largest principal components of  $\Sigma$ , and (b)  $\mathbf{d}$  is also collinear with the PC1s of  $\Sigma_L, \Sigma_H$ . (p. 221).

Both "largest principal components" and "PC1" refer to the eigenvector associated with the largest eigenvalue of  $\Sigma$  and  $Re^p$  denotes  $p$ -dimensional Euclidean space. The stated side-condition to this theorem is that the test correlations are positive in the parent population and remain so in both selected sub-populations.

This theorem is incorrect, because part a and part b of the theorem are incompatible with the selection variable. As shown below, parts a and b do not generally hold in samples created by selection on the basis of a variable  $\Sigma_k \mathbf{y}_k$ .

### *The Effect of Selection on Principal Components*

Like Schönemann, we introduce a  $p$ -dimensional random variable  $\mathbf{y}$  that is distributed  $\mathbf{y} \sim N_p(0, \Sigma)$  in a well-defined parent population (subject subscripts are discarded). A principal component analysis involves the following decomposition of the components of  $\mathbf{y}$  (e.g., Flury, 1988, Chapter 2):

$$(1) \quad \mathbf{y} = \Pi \delta,$$

where  $\mathbf{\Pi}\mathbf{\Pi}^T = \mathbf{I}$ . We call the orthogonal components of the  $q$ -dimensional random vector  $\delta$  principal component (PC) scores and the columns of  $p \times q$  matrix  $\mathbf{\Pi}$ , eigenvectors ( $q \leq p$ ). The PC scores,  $\delta = \mathbf{\Pi}^T \mathbf{y}$ , are distributed  $\delta \sim N_q(\mathbf{0}, \mathbf{\Delta})$ , where the  $q \times q$  covariance matrix  $\mathbf{\Delta}$  is diagonal. The variances of the PC scores, the eigenvalues, are distinct and arranged in descending order,  $\text{diag}(\mathbf{\Delta}) = (\sigma_{\delta_1}^2 > \sigma_{\delta_2}^2 > \dots > \sigma_{\delta_{q-1}}^2 > \sigma_{\delta_q}^2)$ . Finally, we note that

$$(2) \quad \Sigma = \mathbf{\Pi}\mathbf{\Delta}\mathbf{\Pi}^T.$$

We refer to the decomposition in Equation 2 as an eigenvalue decomposition of the covariance matrix  $\Sigma$ .

Following Schönemann, we imagine a selection of individuals from the parent population to form one or more sub-populations on the basis of a linear combination of the components of  $\mathbf{y}$ . The selection variable, denoted  $x$ , equals  $\mathbf{w}^T \mathbf{y}$ , where  $\mathbf{w}$  is a  $p$ -dimensional vector of fixed finite weights. The selection variable is distributed in the parent population  $x \sim N(0, \sigma_x^2)$ , where  $\sigma_x^2$  equals  $\mathbf{w}^T \Sigma \mathbf{w}$ . In Schönemann's (1992) theorem,  $c$  equals zero and  $\mathbf{w}$  equals  $\mathbf{1}$ , the  $p$ -dimensional unit vector. For the subjects in the High set, for example, the following holds:  $\mathbf{1}^T \mathbf{y}$  (or  $\Sigma_k \mathbf{y}_k$ )  $> 0$ .

Given  $\mathbf{y} \sim N_p(\mathbf{0}, \Sigma)$  in the parent population, the covariance and mean structure in selected sub-population are given by the Pearson-Lawley selection formulas (Muthén, 1989):

$$(3) \quad \Sigma_s = \Sigma + \Sigma \mathbf{w} \omega_s \mathbf{w}^T \Sigma$$

$$(4) \quad \mu_s = \Sigma \mathbf{w} \gamma_s,$$

where the subscript  $s$  stands for sub-population. The scalars  $\omega_s$  and  $\gamma_s$  equal

$$(5) \quad \gamma_s = \sigma_x^{-2} (\mu_{xs} - \mu_x) = \sigma_x^{-2} \mu_{xs}$$

$$(6) \quad \omega_s = \sigma_x^{-2} (\sigma_{xs}^2 - \sigma_x^2) \sigma_x^{-2},$$

where  $\mu_{xs}$  and  $\sigma_{xs}^2$  are the mean and variance of the selection variable,  $x$ , in the sub-population. By definition  $\omega_s$  is negative, as selection implies a restriction of range. In terms of the principal components, we have:

$$(7) \quad E(\delta_s) = \mathbf{\Delta} \mathbf{\Pi}^T \mathbf{w} \gamma_s$$

$$(8) \quad \mathbf{\Delta}_s = \mathbf{\Delta} + \mathbf{\Delta} \mathbf{\Pi}^T \mathbf{w} \omega_s \mathbf{w}^T \mathbf{\Pi} \mathbf{\Delta}$$

$$(9) \quad E(\mathbf{y}_s) = \mathbf{\Pi} \mathbf{\Delta} \mathbf{\Pi}^T \mathbf{w} \gamma_s = \mathbf{\Pi} E(\delta_s)$$

$$(10) \quad \Sigma_s = \mathbf{\Pi} (\mathbf{\Delta} + \mathbf{\Delta} \mathbf{\Pi}^T \mathbf{w} \omega_s \mathbf{w}^T \mathbf{\Pi} \mathbf{\Delta}) \mathbf{\Pi}^T = \mathbf{\Pi} \Delta_s \mathbf{\Pi}^T.$$

The mean vectors (Equations 7 and 9) are no longer zero in the selected sub-population. Equation 10 should be read as an eigenvalue decomposition (like Equation 2) in the sub-population if and only if the covariance matrix  $\Delta_s$  is diagonal. As discussed below, this is only the case under special circumstances.

Let  $\pi_i$  represent the eigenvector associated with the  $i^{\text{th}}$  eigenvalue in the parent population, that is, the  $i^{\text{th}}$  column vector in  $\mathbf{\Pi}$ . Let the subscripts  $H$  and  $L$  denote the high and low set, respectively. In the present symbols, part a of Schönemann's theorem states that, given  $\mathbf{w} = \mathbf{1}$ , the vector  $[\mathbf{\Pi} E(\delta_H) - \mathbf{\Pi} E(\delta_L)]$  is collinear with  $\pi_1$ . In view of Equations 9 and 7, the vector  $[\mathbf{\Pi} E(\delta_H) - \mathbf{\Pi} E(\delta_L)]$  equals  $\mathbf{\Pi} \mathbf{\Delta} \mathbf{\Pi}^T \mathbf{w} (\gamma_H - \gamma_L)$ . Now, the required collinearity can only occur if and only if  $\mathbf{w}$  is equal to, or proportional to,  $\pi_1$ :  $\mathbf{w} = \pi_1 \tau$ , where  $\tau$  is a non-zero, finite scalar. As  $\mathbf{\Pi}^T \pi_1$  equals a  $q$ -dimensional vector with unity in position 1 and zeros elsewhere, the vector  $[\mathbf{\Pi} E(\delta_H) - \mathbf{\Pi} E(\delta_L)]$  equals  $\pi_1 \sigma_{\delta_1}^2 \tau (\gamma_H - \gamma_L)$ . Clearly, as  $\sigma_{\delta_1}^2 \tau (\gamma_H - \gamma_L)$  is a scalar,  $\pi_1 \sigma_{\delta_1}^2 \tau (\gamma_H - \gamma_L)$  and  $\pi_1$  are collinear. The problem is that Schönemann specifies  $\mathbf{w} = \mathbf{1}$ , the unit vector, and not  $\mathbf{w} = \pi_1 \tau$ . Part a of Schönemann's theorem cannot therefore be generally true.

Part b of Schönemann's (1992) theorem states that, for  $\mathbf{w} = \mathbf{1}$ , the vector  $\mathbf{\Pi} \mathbf{\Delta} \mathbf{\Pi}^T \mathbf{w} (\gamma_H - \gamma_L)$  is collinear with the eigenvector associated with the largest eigenvalue calculated in the High and Low set. Given part a of the theorem and the fact that eigenvectors have unit length, we can approach part b by asking when these eigenvectors are equal to  $\pi_1$ . Again, only when  $\mathbf{w}$  equals  $\pi_1 \tau$  is this the case. Specifically, the matrix  $(\mathbf{\Delta} \mathbf{\Pi}^T \pi_i \tau \omega_s \tau \pi_i^T \mathbf{\Pi} \mathbf{\Delta})$  is a  $q \times q$  matrix with  $(\sigma_{\delta_i}^4 \tau^2 \omega_s)$  in position  $i, i$  and zero's elsewhere, so the matrix  $\Delta_s$  in Equations 8 and 10 remains diagonal. Because this covariance matrix remains diagonal in the sub-population, Equation 10 can be read as an eigenvalue decomposition and the eigenvectors in the sub-population are equal to those in the parent population. Any other choice of  $\mathbf{w}$ , including  $\mathbf{1}$ , destroys the diagonality of the matrix  $\Delta_s$ , and so destroys the identity of the matrix of eigenvectors in the parent and sub-population(s). Part b of Schönemann's theorem is therefore false.

It appears that Schönemann (1992) is confusing selection based on  $\mathbf{1}^T \mathbf{y}$  and selection based on  $\pi_1^T \mathbf{y}$ . For instance, he suggests (p. 221) that the mean vector of the PC scores in the high set equals  $E(\delta_H)^T = \{[2\sigma_{\delta_1}^2(2\pi)^{-5}], 0, \dots, 0\}$ . Again, this is true if and only if  $\mathbf{w}$  equals  $\pi_1$ . Given this substitution (see Equation 7),  $E(\delta_H)^T$  equals  $[(\sigma_{\delta_1}^2 \gamma_H), 0, \dots, 0]$ , which is equal to

Schönemann's reported mean vector. It is not obvious that  $[2\sigma_{\delta 1}(2\pi)^{-.5}]$  equals  $(\sigma_{\delta 1}^2\gamma_H)$  as we have defined  $\gamma_H$  as  $\sigma_x^{-2}\mu_{xH}$  without providing an expression for  $\mu_{xH}$ . In the present case,  $\sigma_x^2$  equals  $\sigma_{\delta 1}^2$  and  $\mu_{xH}$  equals  $\sigma_{\delta 1}\lambda(0)$ , where, given selection of a high set on the basis of the criterion  $\pi_1^T y > 0$ ,  $\lambda(0)$  equals  $\phi(0)/[1 - \Phi(0)]$ , or  $2(2\pi)^{-.5}$  (e.g., Greene, 1993, p. 685). In the final expression  $\phi(\cdot)$  and  $\Phi(\cdot)$  are the standard normal density and distribution function, respectively. So, in the present case,  $\gamma_H$  equals  $\sigma_{\delta 1}^{-1}2(2\pi)^{-.5}$  and  $(\sigma_{\delta 1}^2\gamma_H)$  equals  $[2\sigma_{\delta 1}(2\pi)^{-.5}]$ .

Finally, Schönemann (1992, p. 223) invokes Perron's theorem apparently in support of part b of his theorem. Perron's theorem (see Basilevsky, 1983, p. 316), gives certain properties of the eigenvalue decomposition of a square matrix that contains only positive elements. The relevance of this to the second part of Schönemann's theorem is hard to see. It is perhaps because Perron's theorem pertains to positive matrices, that Schönemann introduces the condition of positive covariances in the parent and the selected populations. Note that this condition is quite restrictive if  $w$  equals 1.

There is one special case in which the vector  $\pi_1\tau$  can be made to equal 1 by suitable choice of  $\tau$ . In this case, the components of  $y$  are parallel tests, so that the covariance matrix  $\Sigma$  displays a compound symmetric structure. The components of  $\pi_1$  are then equal. However, Jensen's (1992) prescribed test of Spearman's hypothesis cannot be conducted using parallel tests, because their factor loadings do not display systematic variation.

It worth pointing out, finally, that the covariance matrices calculated in groups created by selection on the basis of  $(\pi_1\tau)^T y$  will conform to Flury's common principal component model (Flury, 1988, Chapter 4). This model specifies equal eigenvectors, but varying eigenvalues in the groups under consideration.

### *Discussion*

The importance attached by Jensen to Spearman's hypothesis is that the "hypothesis, if true, would mean that understanding the nature of the statistical black-white differences on various psychometric tests in the cognitive domain depends fundamentally on the nature of  $g$  itself" (Jensen, 1992, p. 229). Within the analysis of principal components, this would mean that the group differences in means and covariance matrices are attributable solely to differences in the principal component that is construed to represent  $g$ . We have seen that such groups can be constructed by explicit selection using the variable  $(\pi_1\tau)^T y$ . Loehlin (1992) has rightly pointed out, however, that this is highly artificial. Americans do not become sorted into the

categories (however fuzzy) of "white" and "black" by an observed selection variable. This type of selection does occur when subjects are selected on the basis of a linear combination of their test scores on an admission test. The effects of such selection on the factor model have been discussed by Muthén (1989). Finally, principal component analysis is not well suited to assess cognitive abilities, because it does not distinguish between factors common to the tests under consideration and factors specific to the tests.

As suggested by Gustafsson (1985, 1992), a useful approach to the investigation of black-white differences in cognitive abilities is by multi-group covariance structure analysis with structured means (Jöreskog & Sörbom, 1989). This approach enables one to test a variety of factor models incorporating general cognitive ability (Gustafsson, 1984; Jensen & Weng, 1994) subject to factorial invariance over the groups. Such models provide a much more comprehensive approach to the investigation of group differences than that provided by Jensen's test of Spearman's hypothesis. The relationship between factor loadings and differences in means is but a single aspect of factorial invariance (Meredith, 1993). In the theory of factorial invariance, the idea, central to Spearman's hypothesis, that within group and between group variation are attributable to the same common factor(s), is expressed in a precise and testable manner. Computer software to implement this theory is readily available (e.g., Jöreskog & Sörbom, 1993; Neale, 1995).

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